Quasiperiodic Solutions of the Fibre Optics Coupled Nonlinear Schrödinger Equations

P L Christiansen[1] , J C Eilbeck[2]
V Z Enolskii[2,3] , N A Kostov[4]
[1] Laboratory of Applied Mathematical Physics
Danish Technical University
Bld. 303, Lyngby 2800, Denmark
[2] Department of Mathematics, Heriot-Watt University
Riccarton, Edinburgh EH14 4AS, UK
[3] Department of Theoretical Physics
Institute of Metal Physics, Vernadsky str. 36
Kiev-680, 252142, Ukraine
[4] Institute of Electronics
Bulgarian Academy of Sciences
Blvd. Tsarigradsko shosse 72, Sofia 178 4, Bulgaria

February 1, 2008

Quasiperiodic Solutions of the Fibre Optics Coupled Nonlinear Schrödinger Equations

P L Christiansen , J C Eilbeck
V Z Enolskii, N A Kostov

Abstract

We consider travelling periodical and quasiperiodical waves in single mode fibres, with weak birefringence and under the action of cross-phase modulation. The problem is reduced to the "1:2:1" integrable case of the two-particle quartic potential. A general approach for finding elliptic solutions is given. New solutions which are associated with two-gap Treibich-Verdier potentials are found. General quasiperiodic solutions are given in terms of two dimensional theta functions with explicit expressions for frequencies in terms of theta constants. The reduction of quasiperiodic solutions to elliptic functions is discussed.

1 Introduction

We consider the system of two coupled nonlinear Schrödinger equations

$$
\begin{aligned}
&iA_Z + \frac{1}{2}A_{TT} + \sigma B + (|A|^2 + \gamma |B|^2)A = 0,\\
&iB_Z + \frac{1}{2}B_{TT} + \sigma A + (|B|^2 + \gamma |A|^2)B = 0.
\end{aligned}
\tag{1.1}
$$

These equations are important for a number of physical applications. For example the dimensionless circularly polarized components A, B of the electric field in a single-mode straight (no twists) fibre satisfy these equations [5, 24]. If the functions A and B do not depend on the variable t, these equations are the generalized discrete self-trapping dimer system [18], which are integrable in terms of elliptic functions. If $\gamma = 0$ these equations are the well known DST dimer equations [13]. The equations (1.1) do not belong to the class of nonlinear evolution equations integrable by means of the inverse scattering method [29]. Nevertheless, they can have exact vector solitons [8, 30], bound solitary waves [7] and periodical solutions [16, 20]. This interesting phenomenon is related to the existence of a fourth integral found by Dowling [9]. Due to this property, two-soliton solutions also exist [30].

In the present paper the general polarization-modulated states – two-gap solutions – are expressed in terms of two-dimensional theta functions. Another motivation for this work is the attempt to classify the exact solutions of the Coupled Nonlinear Schrödinger equation according to the program of classification given in [22].

Let us introduce the new functions $A = (a + ib)/\sqrt{2}$, $B = (a - ib)/\sqrt{2}$ and new independent variables $z = (\gamma + 1)\, Z/2$, $t = \sqrt{\gamma + 1}\, T$ [16]. Then the equations (1.1) can be rewritten as follows

$$
\begin{aligned}
ia_z + a_{tt} + \Omega_0 a + p(|a|^2 + |b|^2)a + q(a^2 + b^2)a^* &= 0,\\
ib_z + b_{tt} - \Omega_0 b + p(|b|^2 + |a|^2)b + q(a^2 + b^2)b^* &= 0,
\end{aligned}
\tag{1.2}
$$

where $\Omega_0 = 2\sigma/3$, $p = \sigma/3$ and $q = 1/3$.

We seek solutions of equations (1.2) in the following form [16]

$$
a(z,t) = q_1(t)\exp(i\Omega z), \quad b(z,t) = q_2(t)\exp(i\Omega z),
\tag{1.3}
$$

where $q_1(t)$ and $q_2(t)$ are real functions, and Ω is an arbitrary real constant. We obtain the system

$$\begin{aligned}
&\frac{d^2}{dt^2}q_1 + (q_1^2 + q_2^2)q_1 = (\Omega - \Omega_0)q_1, \\
&\frac{d^2}{dt^2}q_2 + (q_1^2 + q_2^2)q_2 = (\Omega + \Omega_0)q_2.
\end{aligned} \tag{1.4}$$

These equations describe the known integrable case "1:2:1" of the quartic potential $\mathcal{U} = Aq_1^4 + Bq_1^2q_2^2 + Cq_2^4$ [17], which is separable in ellipsoidal coordinates. The foregoing analysis can be applied also to the integrable potential "1:12:16" which is separable in parabolic coordinates. The dynamics of the other separable cases of the quartic potentials "1:6:1" and "1:6:8" can be expressed in terms of two elliptic functions with different moduli [27].

The paper is organised as follows. In Section 2 we describe the Poisson structure of the integrable system "1:2:1" using the results given in [11, 12]. We derive a Lax representation for the system, prove its complete integrability and construct the separated coordinates. In Section 3 we show that the problem of the description of elliptic solutions for the system (1.4) is reduced to the description of the elliptic potentials for the Schrödinger equation and the construction of the corresponding Lamé polynomials. The problem is closely related to the structure of the locus for the Calogero-Moser system [1]. In Section 4 we analyse the known elliptic solutions for the system (1.4) and show that they are included into the approach developed. Moreover we find a new elliptic solution for the system (1.4) which is associated with the two gap Treibich-Verdier potential [31]. In Section 5 we give an integration of the system via theta functions and calculate the frequencies in terms of theta constants. The question of the periodic solutions is formulated from a theta functional point of view and reduced to the description of some modular varieties given by the vanishing of some two dimensional theta constants. Such a reduction is demonstrated on the example of the elliptic solution associated with the Treibich-Verdier potential.

2 The Lax representation

The system (1.4) is a completely integrable Hamiltonian system with the Hamiltonian

$$H=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{4}(q_1^2+q_2^2)^2-\frac{1}{2}(z_1q_1^2+z_2q_2^2), \tag{2.1}$$

where $(q_i,p_i),\ i=1,2$ are canonical variables with $p_i=\dot{q}_i=dq_i/dx, i=1,2,$ and z_1,z_2 are arbitrary constants.

The Lax representation for the system (1.4) is known in terms of 3×3 matrices [33]. However, we find it convenient to use a 2×2 Lax representation, $\dot{L}(z)=[L(z),M(z)]$ which is a special case of the hierarchy of separable systems discussed in [11, 12].

To do this we fix the following ansatz for the Lax operator.

$$L(z)=\begin{pmatrix} V(z) & U(z)\\ W(z) & -V(z)\end{pmatrix},\quad M(z)=\begin{pmatrix} 0 & 1\\ Q(z) & 0\end{pmatrix}, \tag{2.2}$$

where

$$U(z) = 1-\frac{1}{2}\sum_{i=1}^{2}\frac{q_i^2}{z-z_i},\quad V(z)=-\frac{1}{2}\dot{U}(z), \tag{2.3}$$

$$W(z) = \frac{1}{2}\sum_{i=1}^{2}\frac{p_i^2}{z-z_i}-Q(z),\quad Q(z)=z+\frac{1}{2}\sum_{i=1}^{2}q_i^2. \tag{2.4}$$

The associated algebraic curve $\det(L(z)-yI)=0$ has genus two, and is written as

$$\begin{aligned} w^2 &= -4(z-z_1)(z-z_2)(z^3-z^2(z_1+z_2)+z(z_1z_2-H)-F), \quad (2.5)\\ &= -4\prod_{i=1}^{5}(z-z_i), \end{aligned}$$

where $w=y(z-z_1)(z-z_2)$, H is the Hamiltonian (2.1), F is the additional integral of motion,

$$\begin{aligned} F &= \frac{1}{4}(p_1q_2-p_2q_1)^2+\frac{1}{2}(q_1^2+q_2^2)\left(z_1z_2-\frac{z_2}{2}q_1^2-\frac{z_1}{2}q_2^2\right)\\ &\quad -\frac{1}{2}(z_2p_1^2-z_1p_2^2), \qquad (2.6) \end{aligned}$$

and z_3, z_4, z_5 are the roots of the cubic on the rhs. of (2.5). Let us define new coordinates μ_1, μ_2 as zeros of the function $U(z)$ in the Lax operator, i.e.

$$q_1^2 = -2\frac{(z_1-\mu_1)(z_1-\mu_2)}{z_1-z_2}, \quad q_2^2 = -2\frac{(z_2-\mu_1)(z_2-\mu_2)}{z_2-z_1} \tag{2.7}$$

On can prove (see e.g. [11, 12]) that the canonically conjugated momenta are defined as

$$\pi_i = V(\mu_i) = \frac{w_i}{(\mu_i-z_1)(\mu_i-z_2)}; \tag{2.8}$$

$$\pi_1 = \frac{\dot{\mu}_1}{2}\frac{\mu_1-\mu_2}{(\mu_1-z_1)(\mu_1-z_2)}, \quad \pi_2 = \frac{\dot{\mu}_2}{2}\frac{\mu_2-\mu_1}{(\mu_2-z_1)(\mu_2-z_2)} \tag{2.9}$$

and therefore the coordinates (μ_i, π_i) are the separated coordinates (ellipsoidal coordinates).

It follows from (2.7,2.9) that the the dynamics of the system described in the coordinates (π_i, μ_i) become the Jacobi inversion problem associated with the curve (2.5)

$$\begin{aligned} \int_{\mu_0}^{\mu_1}\frac{d\mu}{w(\mu)} + \int_{\mu_0}^{\mu_2}\frac{d\mu}{w(\mu)} &= a, \\ \int_{\mu_0}^{\mu_1}\frac{\mu d\mu}{w(\mu)} + \int_{\mu_0}^{\mu_2}\frac{\mu d\mu}{w(\mu)} &= 2x+b, \end{aligned} \tag{2.10}$$

where a, b are constants defined by the initial conditions. If $z_{1,2}$ are real and $z_1 > z_2$ then the solution (2.7) is real if $\mu_2 \le z_1 \le \mu_1$ and $\mu_1 \le z_2$ or $z_2 \le \mu_2$.

We remark that the integrable case "1:12:16" of the quartic potential arises in this approach by the introduction of another ansatz for the function U whose zeros define parabolic coordinates [11].

3 Periodic solutions associated with the Lamé equation

In this section we develop a method (see also [10, 15, 19]) which allows us to construct periodic solutions of (1.4) in a straightforward way. The method is based on the application of spectral theory for the Lamé equation with elliptic potentials [1, 23].

$$\frac{d^2}{dx^2}\Psi(x,z) - \mathcal{U}(x)\Psi(x,z) = -z\Psi(x,z). \tag{3.1}$$

with $\mathcal{U}(x)$ being an elliptic potential.

Because the algebraic curve (2.5) associated with the problem has genus two we shall consider a two-gap elliptic potential for the equation (3.1). Such potentials are known to be of the form [1]

$$\mathcal{U}(x) = 2\sum_{i=1}^{N} \wp(x - x_i), \tag{3.2}$$

where $\wp(x)$ is the Weierstrass elliptic functions [2] with the real period $2\omega = 2\omega_1$ and imaginary period $2\omega' = 2\omega_3$. The number N is a positive integer $N > 2$ (the number of "particles") and the numbers $\mathbf{x} = (\tilde{\mu}_1, \dots, x_N)$ belongs to the locus $\mathcal{L}_N$, i.e., the geometrical position of the points given by the equations

$$\mathcal{L}_N = \left\{ (\mathbf{x}); \sum_{i \neq j} \wp'(x_i - x_j) = 0,\ j = 1, \dots N) \right\}. \tag{3.3}$$

Equation (3.1) allows the coalescence of three particles x_i and the potential takes the form [1].

$$\mathcal{U}(x) = 6\sum_{i=1}^{n} \wp(x - x_i) + 2\sum_{j=1}^{m} \wp(x - x_j), \quad 3n + m = N \tag{3.4}$$

The associated algebraic curve of genus two can be described with the help of the Novikov equation [26]. For example, let us consider the two-gap potential for (3.4) normalized by its expansion near the singular point as

$$\mathcal{U}(x) = \frac{6}{x^2} + ax^2 + bx^4 + cx^6 + dx^8 + O(x^{10}). \tag{3.5}$$

Then, following from the Novikov equation [26], the algebraic curve associated with this potential has the form [4]

$$w^2 = z^5 - \frac{35az^3}{2} - \frac{63bz^2}{2} + \left(\frac{567a^2}{8} + \frac{297c}{4} \right) z + \frac{1377ab}{4} - \frac{1287d}{2}. \tag{3.6}$$

Let us consider the *trace formulae* [32] written for the elliptic potential in the form

$$\mu_1 + \mu_2 \quad = \quad -\sum_{j=1}^{N} \wp(x - x_j) + \frac{1}{2}\sum_{j=1}^{5} z_j,$$

$$\mu_1\mu_2 = 3\sum_{i<j}\wp(x-x_i)\wp(x-x_j) - \frac{Ng_2}{8} + \frac{1}{2}\sum_{i<j} z_i z_j - \frac{3}{8}\left(\sum_{j=1}^{5} z_j\right)^2 \tag{3.7}$$

We require that the eigenvalues z_1 and z_2 be the branching points of the curve associated with the two-gap elliptic potential. Using the relation which follows from (2.7)

$$q_1^2 + q_2^2 = 2(\mu_1 + \mu_2 - z_1 - z_2), \tag{3.8}$$

then from the first trace formula we can write (1.4) in the form

$$\frac{d^2}{dx^2}q_1 - \mathcal{U}(x)q_1 = (\Omega - \Omega_0 + 2z_1 + 2z_2)q_1 \tag{3.9}$$

$$\frac{d^2}{dx^2}q_2 - \mathcal{U}(x)q_2 = (\Omega + \Omega_0 + 2z_1 + 2z_2)q_2, \tag{3.10}$$

where we set without loss of generality $\sum_{j=1}^{5} z_j = 0$.

Then if the relations

$$\Omega - \Omega_0 = -3z_1 - 2z_2, \quad \Omega + \Omega_0 = -2z_1 - 3z_2. \tag{3.11}$$

hold, the problem of finding elliptic solutions for (1.4) is reduced to the calculation of two Lamé polynomials, i.e. the values of the eigenfunctions of (2.2) corresponding to the spectral parameter fixed at the ends of the gaps

$$q_1 = C_1\Psi(x, z_1), \quad q_2 = C_2\Psi(x, z_2), \tag{3.12}$$

where C_i are some constants, corresponding to a point $\mathbf{x}$ on the locus $\mathcal{L}_N$.

For the elliptic potential without degeneracy (3.2) we have from (2.7,3.7) and (3.8) the general formula for elliptic solutions of equations (1.4).

$$q_1^2 = \frac{1}{z_2 - z_1}\left(2z_1^2 + 2z_1\sum_{i=1}^{N}\wp(x-x_i) + 6\sum_{1\le i<j\le N}\wp(x-x_i)\wp(x-x_j) - \frac{Ng_2}{4} + \sum_{1\le i<j\le 5} z_i z_j\right), \tag{3.13}$$

$$q_2^2 = \frac{1}{z_1 - z_2}\left(2z_2^2 + 2z_2\sum_{i=1}^{N}\wp(x-x_i) + 6\sum_{1\le i<j\le N}\wp(x-x_i)\wp(x-x_j) - \frac{Ng_2}{4} + \sum_{1\le i<j\le 5} z_i z_j\right). \tag{3.14}$$

A problem with the application of this formulae is that we have to find points $\mathbf{x}$ on the locus such that the functions (3.13,3.14) are real and finite. Some special cases are known, for example the analytical description of the locus (3.3) for two-gap elliptic potentials for which x_i are half-periods, $x_i = \omega_i$ (Treibich-Verdier potentials, [31]). Details are given in [14]. The curves and Lamé polynomial associated with these potentials are given in Table 1.

4 Elliptic solitary waves

We shall show in this Section that the known periodic solutions of (1.4) are associated with one and two-gap elliptic potentials of the Schrödinger equation.

4.1 One-gap potentials

The equations (1.4) have the following solutions [7]:

$$q_1 = C_1 \mathrm{sn}(\alpha x, k), \quad q_2 = C_2 \mathrm{cn}(\alpha x, k), \tag{4.1}$$

where the amplitudes C_1 ,C_2, the modulus k of the elliptic functions, and the temporal pulsewidth $1/\alpha$ of the waves are defined in terms of the physical parameters Ω and Ω_0 as

$$\alpha^2 k^2 = 2\Omega_0,\ C_1^2 = \Omega - 3\Omega_0 + \alpha^2,\ C_2^2 = \Omega + \Omega_0 + \alpha^2.$$

The parameters can be expressed in terms of the Weierstrass parameters $e_3 \leq e_2 \leq e_1$ as follows, $\Omega = -(e_2 + e_3)/2,\ \Omega_0 = (e_2 - e_3)/2,\ \alpha = \sqrt{e_1 - e_3},\ C_1 = \sqrt{e_1 - 2e_2},\ C_2 = \sqrt{e_1 - 2e_3}$

The solutions (4.1) are associated with the eigenvalues $z_1 = e_2$ and $z_2 = e_3$ of the one-gap Lamé potential. In this case $H = \frac{7}{4}(e_2 + e_3)^2 + e_2 e_2,\ F = \frac{1}{2}(e_2 + e_3)^2$ and the curve (2.5) reduces (after a transformation of the spectral parameter $z \to -z + e_2 + e_3$) to the product of the Weierstrass cubic and a perfect square

$$w^2 = 4(z - e_1)(z - e_2)(z - e_3)\left(z - \frac{3}{2}e_1\right)^2. \tag{4.2}$$

It is easy to see that one of the variables μ_i is pinched at the point $3e_1/2$ and the Jacobi inversion problem written for the curve (4.2) becomes the inversion of the elliptic integral for the remaining variable μ.

Table 1: The algebraic curves and Lamé polynomials

$\mathcal{U}_N$	The curves and Lame polynomials $\Lambda(x)$
$\mathcal{U}_3$	$\mathcal{U}(x) = 6\wp(x)$ $w^2 = -(z^2 - 3g_2)\prod_{i=1}^{3}(z - 3e_i)$ $\Lambda_{ij} = \sqrt{(\wp(x) - e_i)(\wp(x) - e_j)}, \quad (z = 3e_k), \quad i \neq j \neq k = 1, 2, 3$ $\Lambda_{\pm} = \wp(x) \pm \frac{1}{2}\sqrt{\frac{g_2}{3}}, (z = \pm\sqrt{3g_2})$
$\mathcal{U}_4$	$\mathcal{U}(x) = 6\wp(x) + 2\wp(x + \omega_i)$ $w^2 = -(z + 6e_i)\prod_{k=1}^{4}(z - z_k(i)), \quad i = 1, 2, 3$ $z_{1,2}(i) = e_j + 2e_i \pm 2\sqrt{(e_i - e_j)(2e_j + 7e_i)}$ $z_{3,4}(i) = e_k + 2e_i \pm 2\sqrt{(e_i - e_k)(2e_k + 7e_i)}$ $\Lambda_{ik} = \sqrt{(\wp(x) - e_i)(\wp(x) - e_j)} + \frac{1}{3}[(e_i - e_k) \pm \sqrt{(e_i - e_k)(7e_i + 2e_k)}]\sqrt{\frac{\wp(x) - e_j}{\wp(x) - e_i}}$ $(z = e_k + 2e_i \pm 2\sqrt{(e_i - e_k)(2e_k + 7e_i)})$ $\Lambda_0 = \wp(x) - e_i \ (z = -6e_i)$
$\mathcal{U}_5$	$\mathcal{U}(x) = 6\wp(x) + 2\wp(x + \omega_i) + 2\wp(x + \omega_j)$ $w^2 = \prod_{i=1}^{i=5}(z - z_i(k)), \quad j = 1, 2, 3$ $z_4(k) = 6e_k - 3e_i, \ z_5(k) = 6e_i - 3e_k$ $\prod_{i=1}^{3}(z - z_i(k)) = z^3 - 3z^2 e_k + (51e_k^2 - 20g_2)z - 369e_k^3 + 132e_k g_2$ $\Lambda_i = \sqrt{(\wp(x) - e_i)(\wp(x) - e_k)} + (e_i - e_j)\sqrt{\frac{\wp(x) - e_k}{\wp(x) - e_i}}, \quad (z = 6e_i - 3e_j)$ $\Lambda_j = \sqrt{(\wp(x) - e_j)(\wp(x) - e_k)} + (e_j - e_i)\sqrt{\frac{\wp(x) - e_k}{\wp(x) - e_j}}, \quad (z = 6e_j - 3e_i)$ $\Lambda_n = \sqrt{(\wp(x) - e_i)(\wp(x) - e_j)} + \tilde{a}_{ij}\sqrt{\frac{\wp(x) - e_i}{\wp(x) - e_j}} + \tilde{a}_{ji}\sqrt{\frac{\wp(x) - e_j}{\wp(x) - e_i}},$ $(z = z_n), i \neq j \neq k, \ n = 1, 2, 3, \tilde{a}_{ij} = \frac{15e_i^2 + 27e_j^2 - 6e_i e_j - z_n^2 + 2z_n(e_j - e_i)}{24(e_j - e_i)}$

4.2 Periodic solutions associated with the two-gap Lamé potential

It was shown in [16] by direct substitution that the equations (1.4) have the following periodical solutions:

$$q_1 = C\mathrm{dn}(\alpha x, k)\mathrm{sn}(\alpha x, k), \quad q_2 = C\mathrm{dn}(\alpha x, k)\mathrm{cn}(\alpha x, k), \tag{4.3}$$

where sn, cn, dn are the standard Jacobian elliptic functions [2], k is the modulus of the elliptic functions $0 < k < 1$, and the characteristic parameters of the wave: amplitude C, temporal pulsewidth $1/\alpha$ and k are related to the physical parameters Ω and, Ω_0 through the following dispersion relations

$$C^2 = \frac{6\Omega}{5} + 2\Omega_0, \quad k^2 = \frac{10\Omega_0}{3\Omega + 5\Omega_0}, \quad \alpha^2 = \frac{3\Omega + 5\Omega_0}{15}. \tag{4.4}$$

Another elliptic solution was found in [20].

$$q_1 = C_1\alpha k\mathrm{cn}(\alpha x, k)\mathrm{sn}(\alpha x, k), q_2 = C_2\mathrm{sn}^2(\alpha x, k) + C_3, \tag{4.5}$$

where C, C_1, α and k are expressed through physical parameters Ω and Ω_0 by the following relations

$$\begin{aligned} C_1^2 &= \frac{6c}{5}, \quad C_3 = -\left(1 + \frac{c}{5}\right)\sqrt{\Omega_0}, \quad C_2^2 = \frac{12\Omega}{5}, \\ \alpha^2 &= \frac{\Omega_0}{2} + \frac{c\Omega_0}{15} - \frac{\Omega}{10}, \quad k^2 = -\frac{20c}{(c-15)(c+5)}, \end{aligned} \tag{4.6}$$

where $c^2 = 15\Omega/\Omega_0$.

The solutions (4.3,4.5) can be obtained from the general periodic solution (3.13,3.14). The solutions (4.3) have the following spectral interpretation. They are the Lamé polynomials associated with the eigenvalues $3e_2$, $3e_3$ (see Table 1),

$$\begin{aligned} q_1^2 &= -\frac{6}{e_1 - e_3}(\wp(x+\omega_3) - e_1)(\wp(x+\omega_3) - e_3) \\ q_2^2 &= \frac{6}{e_1 - e_3}(\wp(x+\omega_3) - e_1)(\wp(x+\omega_3) - e_2). \end{aligned}$$

One can show that the relations (3.11) become (4.4). Analogously one can prove that the normalised Lamé polynomials associated with the eigenvalues $3e_2$, $-\sqrt{3g_2}$ gives the solution (4.5). We prolong this analogue by constructing some new periodic solutions corresponding to the Treibich-Verdier potentials given in Table 1.

4.3 Periodic solutions associated with the two-gap Treibich-Verdier potentials

Below we construct the two periodic solutions associated with the Treibich-Verdier potential $\mathcal{U}_3$ in the same way as the solutions (4.3) and (4.5) are connected with the two-gap Lamé potential $\mathcal{U}_4$. Let us consider the potential

$$\mathcal{U}_4(x) = 6\wp(x+\omega_3) + 2\frac{(e_1-e_2)(e_1-e_3)}{\wp(x+\omega_3)-e_1} \tag{4.7}$$

and construct the solution in terms of Lamé polynomials associated with the eigenvalues z_1, z_2, $z_1 > z_2$

$$\begin{aligned} z_1 &= e_2 + 2e_1 + 2\sqrt{(e_1-e_2)(7e_1+2e_2)}, \\ z_2 &= e_3 + 2e_1 + 2\sqrt{(e_1-e_3)(7e_1+2e_3)}. \end{aligned} \tag{4.8}$$

The finite and real solutions q_1, q_2 have the form

$$\begin{aligned} q_1 &= iC\sqrt{\wp(x+\omega_3)-e_3}\left(\sqrt{(\wp(x+\omega_3)-e_1}\right. \\ &\qquad \left. +\frac{e_1-e_2+\sqrt{(e_1-e_2)(7e_1+2e_2)}}{3\sqrt{(\wp(x+\omega_3)-e_1}}\right), \end{aligned} \tag{4.9}$$

$$\begin{aligned} q_2 &= C\sqrt{\wp(x+\omega_3)-e_2}\left(\sqrt{(\wp(x+\omega_3)-e_1}\right. \\ &\qquad \left. +\frac{e_1-e_3+\sqrt{(e_1-e_3)(7e_1+2e_3)}}{3\sqrt{(\wp(x+\omega_3)-e_1}}\right). \end{aligned} \tag{4.10}$$

Here ω_3 is the pure imaginary half period, $0 \le t \le 2\omega$ with 2ω being the real period, and

$$C^2 = \frac{18}{z_1 - z_2} > 0. \tag{4.11}$$

The parameters Ω and Ω_0 are linked with the Weierstrass parameters e_i through

$$2\Omega = -5(z_1+z_2), \quad 2\Omega_0 = z_1 - z_2 \tag{4.12}$$

with z_1, z_2 given by (4.8). By eliminating e_i from these formula and from the formula for the amplitude (4.11) we arrive at the dispersion relations,

$$C^2\Omega_0 = 9, \quad 0 \le k^2 \le 1,$$

$$\frac{\Omega}{5\Omega_0} = \frac{k^2 - 2 - 2\sqrt{1-k^2}\sqrt{4-k^2} - 2\sqrt{4-3k^2}}{k^2 + 2\sqrt{1-k^2}\sqrt{4-k^2} - 2\sqrt{4-3k^2}}, \quad \Omega_0 \le \frac{\Omega}{15}. \tag{4.13}$$

Analogously we can find the elliptic solution associated with the eigenvalues

$$z_1 = e_2 + 2e_1 + 2\sqrt{(e_1 - e_2)(7e_1 + 2e_2)}, \quad z_2 = -6e_1, \tag{4.14}$$

We have

$$q_1 = C(\wp(x+\omega_3) - e_1), \tag{4.15}$$

$$q_2 = iC\sqrt{\wp(x+\omega_3) - e_3}\left(\sqrt{(\wp(x+\omega_3) - e_1} + \frac{e_1 - e_2 + \sqrt{(e_1-e_2)(7e_1+2e_2)}}{3\sqrt{(\wp(x+\omega_3) - e_1}}\right), \tag{4.16}$$

where C is given by (4.11) but with z_1, z_2 given by (4.14). The corresponding dispersion relation has the form

$$C^2\Omega_0 = 9, \quad \frac{7}{8} \le k^2 \le 1,$$

$$\frac{\Omega}{5\Omega_0} = \frac{3 - 2k^2 - 2\sqrt{1-k^2}\sqrt{4-k^2}}{5 - 2k^2 + 2\sqrt{1-k^2}\sqrt{4-k^2}}, \quad 0 \le \frac{\Omega}{5\Omega_0} \le \frac{1}{3}. \tag{4.17}$$

The Treibich-Verdier potential $\mathcal{U}_5$ also yields elliptic solutions, but the solutions corresponding to the potential $\mathcal{U}_5$ given in the Table blow up. From general considerations we conjecture that there exist non blow-up real solutions associated with the isospectral deformation of this potential.

The solutions derived represent stationary periodical waves. However, taking into account the invariance of the equations (1.1) under a Galilean transformation [16, 9], they also represent travelling periodical waves.

5 Exact quasi-periodic solutions

In this section we give the theta functional expressions for the trajectories of the system under consideration using the Rosenhain ultraelliptic functions [28] (see also [25]), i.e. Abelian functions associated with an algebraic curve of genus two. We also show how to reduce ultraelliptic solutions to the elliptic solutions discussed before.

5.1 Theta functional integration

Let $(z_\alpha, z_\beta) \in (z_1, \ldots, z_5)$ be two arbitrary branching points of the curve (2.5). After the transformation $(w, z) \to (\zeta, \xi)$,

$$\begin{aligned} w &= 2i(z_\beta - z_\alpha)\sqrt{(z_i - z_\alpha)(z_j - z_\alpha)(z_k - z_\alpha)}\zeta, \\ z &= (z_\beta - z_\alpha)\xi + z_\alpha \end{aligned} \tag{5.1}$$

the curve becomes [28]

$$\zeta^2 = \xi(1-\xi)(1-\kappa^2\xi)(1-\lambda^2\xi)(1-\mu^2\xi) \tag{5.2}$$

with

$$\kappa^2 = \frac{z_\beta - z_\alpha}{z_i - z_\alpha},\ \lambda^2 = \frac{z_\beta - z_\alpha}{z_j - z_\alpha},\ \mu^2 = \frac{z_\beta - z_\alpha}{z_k - z_\alpha}. \tag{5.3}$$

Let us fix on (2.5) the homology basis, $(\mathbf{A}; \mathbf{B}) = (A_1, A_2; B_1, B_2)$ and the conjugated basis of differentials of the first kind

$$\mathbf{v} = (v_1, v_2), \quad v_1 = \frac{c_{11}\xi + c_{12}}{\zeta} d\xi \quad v_2 = \frac{c_{21}\xi + c_{22}}{\zeta} d\xi$$

normalised as

$$\left(\oint_{A_1} \mathbf{v}, \oint_{A_2} \mathbf{v}; \oint_{B_1} \mathbf{v}, \oint_{B_2} \mathbf{v} \right) = (\mathbf{1}_2; \tau), \tag{5.4}$$

where $\mathbf{1}_2$ is the 2×2 unit matrix and the period matrix τ belongs to the Siegel upper half space $\mathcal{S}_2$ of degree 2. The Jacobi inversion problem can be rewritten as

$$\int_{\mu_0}^{\mu_1} v_1 + \int_{\mu_0}^{\mu_2} v_1 = 2c_{11}\sqrt{\frac{z_\beta - z_\alpha}{\kappa\lambda\mu}}x = u_1 x + u_{10}, \tag{5.5}$$

$$\int_{\mu_0}^{\mu_1} v_2 + \int_{\mu_0}^{\mu_2} v_2 = 2c_{21}\sqrt{\frac{z_\beta - z_\alpha}{\kappa\lambda\mu}}x = u_2 x + u_{20} \tag{5.6}$$

The solution of the problem is expessed in terms of theta functions with characteristics $[\varepsilon]$ defined on $C^2 \times \mathcal{S}_2$

$$\theta\left[\begin{smallmatrix}\varepsilon' \\ \varepsilon''\end{smallmatrix}\right](\mathbf{w}|\tau) = \sum_{n \in \mathbf{Z}^2} \exp\left\{ \pi i \left\langle (n + \frac{\varepsilon'}{2})\tau, (n + \frac{\varepsilon'}{2}) \right\rangle + 2\pi i \left\langle n + \frac{\varepsilon'}{2}, \mathbf{w} + \frac{\varepsilon''}{2} \right\rangle \right\}, \tag{5.7}$$

where $\langle , \rangle$ is a Euclidean scalar product and Im τ is positive definite. For integer characteristics we have

$$\begin{aligned}\theta\left[\begin{smallmatrix}\varepsilon'\\ \varepsilon''\end{smallmatrix}\right](\mathbf{w}|\tau) &= \exp\pi\, i\,\left[\frac{1}{4}\langle\varepsilon',\tau\varepsilon'\rangle+\langle\varepsilon'\mathbf{w}\rangle+\frac{1}{2}\langle\varepsilon',\varepsilon''\rangle\right]\\ &\times\ \theta\left(\mathbf{w}+I\frac{\varepsilon''}{2}+\tau\frac{\varepsilon'}{2}|\tau\right),\end{aligned} \tag{5.8}$$

where $\theta(z|\tau)=\theta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](z|\tau)$. The characteristic $[\varepsilon]$ is called *even* if the corresponding theta function is an even function ($\langle\varepsilon,\varepsilon'\rangle=0\,(\mathrm{mod}2)$) and *odd* if the corresponding theta function is an odd function ($\langle\varepsilon,\varepsilon'\rangle=1\,(\mathrm{mod}2)$).

The values of theta functions and their derivatives at the point $z=0$ are called theta constants which we denote by $\theta[\varepsilon](0;\tau)=\theta[\varepsilon]$, if the characteristic $[\varepsilon]$ is even and $\partial\theta[\varepsilon](z;\tau)/\partial z_i|_{z=0}=\theta_i[\varepsilon], i=1,2$ if the chacteristic $[\varepsilon]$ is odd.

The function (5.7) satisfies two sets of functional equations [25], the *transformational property*

$$\begin{aligned}&\theta[\varepsilon](\mathbf{w}+\mathbf{n}''+\mathbf{n}'\tau|\tau)\\ &=\exp\pi i\Big[-\langle\mathbf{n}'\tau,\mathbf{n}'\rangle-2\langle\mathbf{n}'',\mathbf{w}\rangle+\langle\varepsilon',\mathbf{n}'\rangle-\langle\varepsilon'',\mathbf{n}'\rangle\Big]\,\theta[\varepsilon](\mathbf{w}|\tau)\end{aligned} \tag{5.9}$$

where $\mathbf{n}',\mathbf{n}''\in\mathbf{Z}^g$ and the *modular property*, which describes the transformation of the theta function under the action of the group $Sp_4(\mathbf{Z})$.

The branching points are expressed in terms of theta constants as

$$\kappa^2=\frac{\theta^2\left[\begin{smallmatrix}11\\ 00\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}10\\ 00\end{smallmatrix}\right]}{\theta^2\left[\begin{smallmatrix}01\\ 00\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}00\\ 00\end{smallmatrix}\right]},\quad \lambda^2=\frac{\theta^2\left[\begin{smallmatrix}10\\ 01\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}11\\ 00\end{smallmatrix}\right]}{\theta^2\left[\begin{smallmatrix}00\\ 01\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}01\\ 00\end{smallmatrix}\right]},\quad \mu^2=\frac{\theta^2\left[\begin{smallmatrix}10\\ 01\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}10\\ 00\end{smallmatrix}\right]}{\theta^2\left[\begin{smallmatrix}00\\ 01\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}00\\ 00\end{smallmatrix}\right]}.$$

The Jacobi inversion problem has the solution

$$\frac{\theta^2\left[\begin{smallmatrix}10\\ 11\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}{\theta^2\left[\begin{smallmatrix}00\\ 11\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}=-\kappa\lambda\mu\;\tilde{\mu}_1\tilde{\mu}_2, \tag{5.10}$$

$$\frac{\theta^2\left[\begin{smallmatrix}10\\ 01\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}{\theta^2\left[\begin{smallmatrix}00\\ 11\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}=-\frac{\kappa\lambda\mu\;(1-\tilde{\mu}_1)(1-\tilde{\mu}_2),}{\sqrt{1-\kappa^2}\sqrt{1-\lambda^2}\sqrt{1-\mu^2}} \tag{5.11}$$

$$\frac{\theta^2\left[\begin{smallmatrix}01\\ 01\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}{\theta^2\left[\begin{smallmatrix}00\\ 11\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}=\frac{\lambda\mu\;(1-\kappa^2\tilde{\mu}_1)(1-\kappa^2\tilde{\mu}_2)}{\sqrt{1-\kappa^2}\sqrt{\kappa^2-\lambda^2}\sqrt{\kappa^2-\mu^2}}, \tag{5.12}$$

$$\frac{\theta^2\left[\begin{smallmatrix}01\\ 00\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}{\theta^2\left[\begin{smallmatrix}00\\ 11\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}=\frac{\kappa\mu\;(1-\lambda^2\tilde{\mu}_1)(1-\lambda^2\tilde{\mu}_2)}{\sqrt{1-\lambda^2}\sqrt{\lambda^2-\mu^2}\sqrt{\kappa^2-\lambda^2}}, \tag{5.13}$$

$$\frac{\theta^2\left[\begin{smallmatrix}00\\ 00\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}{\theta^2\left[\begin{smallmatrix}00\\ 11\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}=\frac{\kappa\lambda\;(1-\mu^2\tilde{\mu}_1)(1-\mu^2\tilde{\mu}_2)}{\sqrt{1-\mu^2}\sqrt{\kappa^2-\mu^2}\sqrt{\lambda^2-\mu^2}}. \tag{5.14}$$

The solutions of equations (1.4) are expressed in terms of the formulae (5.10-5.14). To do this we have to put z_1, z_2 into correspondence with two branching points of the curve (5.2). In particular, let us choose $z_{1,2}$ in such a way that their images $\tilde{z}_{1,2}$ under the transformation (5.1) becomes the branching points 0 and λ^2 and the images $\tilde{\mu}_{1,2}$ of $\mu_{1,2}$ move inside the gaps $\tilde{z}_2 = 0 \leq \tilde{\mu}_2 \leq \mu^2$, $\tilde{z}_1 = \lambda^2 \leq \tilde{\mu}_1 \leq 1$. Let us also fix the vector $\mathbf{u}_0$ as a half period $\mathbf{u}_0 = (1/2)(\tau_{11} + \tau_{12}, \tau_{12} + \tau_{22} + 1)$, which shifts the characteristics at $\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right]$. Then we find the following quasiperiodic solutions

$$\begin{aligned} q_1^2 &= -2\frac{(z_\beta - z_\alpha)^2\sqrt{1-\lambda^2}\sqrt{\lambda^2-\mu^2}\sqrt{\kappa^2-\lambda^2}}{(z_1-z_2)\kappa\mu} \\ &\quad\times\frac{\theta^2\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}{\theta^2\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)} \end{aligned} \tag{5.15}$$

$$q_2^2 = -2\frac{(z_\beta - z_\alpha)^2}{(z_1-z_2)\kappa\lambda\mu}\frac{\theta^2\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}{\theta^2\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right](\mathbf{u}(x-x_0)+\mathbf{u}_0|\tau)}, \tag{5.16}$$

where the frequences $u_{1,2}$ are in general noncommensurable,

$$u_1 = \frac{\sqrt{z_\beta - z_\alpha}\,\theta_2\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right]}{\pi^2\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]}, \quad u_2 = -\frac{\sqrt{z_\beta - z_\alpha}\,\theta_1\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right]}{\pi^2\theta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]}, \tag{5.17}$$

We emphasise that the solution given is parametrised only by the the τ-matrix and expressed in terms of theta functions and theta constants which are rapidly convergent and therefore this expression is convenient for numerical calculations.

5.2 Reduction to elliptic functions

The given solutions are quasiperiodical functions of time. They are reduced to elliptic functions under some restrictions on the parameters of the system which can be formulated in terms of conditions on the matrix τ.

More precisely, it follows from the explicit formulae (5.15,5.16) for q_1, q_2 that the following two conditions are sufficient to require the solution be an elliptic function of x.

1. The matrix τ is taken in the form

$$\tau = \begin{pmatrix} \tau_{11} & \frac{1}{N} \\ \frac{1}{N} & \tau_{22} \end{pmatrix}, \quad N \in \mathbf{N} \tag{5.18}$$

2. u_1 or $u_2 = 0$.

It follows immediately from the transformational properties of theta functions that under these conditions the solution is an elliptic function of the N-th order. The condition 2 is reduced with the help of the formulae (5.17) to the condition of the vanishing of some theta constants. It is remarkable that the conditions 1, 2 are also sufficient conditions, as was proved in [14] within the Weierstrass reduction theory of theta functions to lower genera [3, 21]. If only the condition 1 is satisfied then the solution (5.15,5.16) is quasiperiodic and expressed in term of two elliptic functions with the Jacobian moduli $N\tau_{11}$ and $N\tau_{22}$.

To demonstrate how this approach works we derive the elliptic solution (4.9,4.10) from the theta functional approach. To this end we fix the period matrix in the form (5.18) and put $N = 4$. We remark that this case of the reduction of theta functions was studied by Bolza [6] (see also [3]).

The computation is based on the addition theorem of the second order (see e.g. [25])

$$\begin{aligned} &\theta[\varepsilon](\mathbf{x}|\tau)\theta[\delta](\mathbf{y}|\tau) \qquad (5.19)\\ =\ &\sum_{\rho}\theta\left[\begin{array}{c}\frac{1}{2}(\varepsilon'+\delta')+\rho\\ \varepsilon''+\delta''\end{array}\right](\mathbf{x}+\mathbf{y}|2\tau)\theta\left[\begin{array}{c}\frac{1}{2}(\varepsilon'-\delta')+\rho\\ \varepsilon''-\delta''\end{array}\right](\mathbf{x}-\mathbf{y}|2\tau),\end{aligned}$$

where the summation runs over $\rho = (0,0), (0,1), (1,0), (1,1)$ and

$$\begin{aligned}&\theta\left[\begin{array}{cc}\varepsilon_1' & \varepsilon_2'\\ \varepsilon_1'' & \varepsilon_2''\end{array}\right]\left(\mathbf{z}\middle|\left(\begin{array}{cc}\tau & 1\\ 1 & \tilde{\tau}\end{array}\right)\right)\\ &= \mathrm{e}^{-\frac{1}{2}\pi i\varepsilon_1'\varepsilon_2'}\theta\left[\begin{array}{cc}\varepsilon_1' & \varepsilon_2'\\ \varepsilon_1''+\varepsilon_2' & \varepsilon_2''+\varepsilon_1'\end{array}\right]\left(\mathbf{z}\middle|\left(\begin{array}{cc}\tau & 0\\ 0 & \tilde{\tau}\end{array}\right)\right) \qquad (5.20)\end{aligned}$$

We now show that under conditions 1^0 and 2^0 the argument of the theta function in (5.10-5.14) becomes $(x/2\omega, 0)$. The condition $u_2 = 0$ or $\theta_1\left[{}^{11}_{01}\right] = 0$ becomes the condition (see Appendix)

$$\frac{\sqrt{2}\vartheta_2^2\widetilde{\vartheta}_3}{\vartheta_3} + \sqrt{\vartheta_3^2\widetilde{\vartheta}_3^2 + \vartheta_2^2\widetilde{\vartheta}_4^2 - \vartheta_4^2\widetilde{\vartheta}_2^2} = 0, \qquad (5.21)$$

where $\vartheta_i = \vartheta_i(0; 4\tau_{11})$, $\widetilde{\vartheta}_i = \vartheta_i(0; 4\tau_{22})$, $i = 2, 3, 4$. The condition (5.21) is equivalent to the relations between Jacobi theta constants

$$\tilde{\vartheta}_2^2 = \tilde{\vartheta}_3^2\frac{\vartheta_4^2}{\vartheta_3^2}\left(1 - 4\frac{\vartheta_2^4}{\vartheta_3^4}\right), \ \tilde{\vartheta}_4^2 = \tilde{\vartheta}_3^2\frac{\vartheta_2^2}{\vartheta_3^2}\left(1 - 4\frac{\vartheta_4^4}{\vartheta_3^4}\right). \qquad (5.22)$$

Using the explicit expressions for the branching points (see Table 1) we find after substitution of the corresponding theta constants (see Appendix) in (5.17)

$$u_1 = \frac{\sqrt{z_\beta - z_\alpha}\vartheta_3}{2\pi\vartheta_2\sqrt{4\vartheta_2^4 - 3\vartheta_3^4}}.$$

Chosing $z_{\beta,\alpha} = e_3 + 2e_1 \pm 2\sqrt{(e_1 - e_3)(7e_1 + 2e_3)}$ we find after the transformation $E = BAB$ (see e.g.[2]), $u_1 = 1/2\omega = \sqrt{e_1 - e_3}/\pi\vartheta_3^2$, in accordance with [2].

The same arguments permit us to compute (5.15,5.16) in terms of elliptic functions. Let us consider the theta function entering into the expression for (5.15). Applying the addition theorem (5.19) and putting $w_2 = 0$ according to condition 2^0 and using the expressions for theta constants we find

$$\begin{aligned}
&\frac{\theta\left[{}^{01}_{01}\right]\left(\frac{x-x_0}{2\omega}, 0|\tau\right)}{\theta\left[{}^{11}_{01}\right]\left(\frac{x-x_0}{2\omega}, 0|\tau\right)} = C_1 \frac{\vartheta_2\left(\frac{x-x_0}{2\omega}\right)}{\vartheta_3\left(\frac{x-x_0}{2\omega}\right)\vartheta_1^2\left(\frac{x-x_0}{2\omega}\right)} &\quad (5.23)\\
&\times\left[3\vartheta_4^2\vartheta_3^2\left(\frac{x-x_0}{2\omega}\right) + \vartheta_2^2\vartheta_1^2\left(\frac{x-x_0}{2\omega}\right) + \sqrt{\vartheta_3^4 - 4\vartheta_4^4}\vartheta_1^2\left(\frac{x-x_0}{2\omega}\right)\right]\\
&\frac{\theta\left[{}^{10}_{10}\right]\left(\frac{x-x_0}{2\omega}, 0|\tau\right)}{\theta\left[{}^{11}_{01}\right]\left(\frac{x-x_0}{2\omega}, 0|\tau\right)} = C_2 \frac{\vartheta_4\left(\frac{x-x_0}{2\omega}\right)}{\vartheta_3\left(\frac{x-x_0}{2\omega}\right)\vartheta_1^2\left(\frac{x-x_0}{2\omega}\right)} &\quad (5.24)\\
&\times\left[3\vartheta_2^2\vartheta_3^2\left(\frac{x-x_0}{2\omega}\right) - \vartheta_4^2\vartheta_1^2\left(\frac{x-x_0}{2\omega}\right) + \sqrt{\vartheta_3^4 - 4\vartheta_2^4}\vartheta_1^2\left(\frac{x-x_0}{2\omega}\right)\right],
\end{aligned}$$

where $C_{1,2}$ can be computed using the theta constants from the Appendix,

$$\begin{aligned}
C_2 &= \frac{i\sqrt{2}}{2}\frac{\vartheta_2^{5/2}}{\sqrt[4]{s}\sqrt{\vartheta_2^4 - \vartheta_4^4 + \vartheta_2^2 s(\vartheta_2^2 + s)}}, &\quad (5.25)\\
C_1 &= \frac{\sqrt{2}}{2}(-1)^{1/4}\frac{\vartheta_2^4}{\vartheta_4^{3/2}\sqrt[4]{t}\sqrt{\vartheta_2^4 - \vartheta_4^4 - \vartheta_4^2 t(t - \vartheta_4^2)}},
\end{aligned}$$

where $s = \sqrt{\vartheta_3^4 - 4\vartheta_4^4}, t = \sqrt{\vartheta_3^4 - 4\vartheta_2^4}$ One can see that substituting (5.23, 5.24) and (3.13,3.14) performing the transformation BAB (see, e.g. [2]) and setting $x_0 = \omega_3$ we obtain the elliptic solutions (4.9,4.10). In the same way the another elliptic solution associated with two-gap Treibich-Verdier potential (4.15,4.16) can be obtained from the theta functional solutions.

6 Acknowledgements

One of the authors (NAK) acknowledges support from EEC grant ERB-CIPA-CT-92-0473 the other (VZE) acknowledges support from the Royal Society under an ex-quota fellowship grant.

A Theta Constants

We denote the Jacobi theta constants by $\vartheta_j = \vartheta_j\,(0|2^p\tau_{11})$, $\widetilde{\vartheta}_j = \vartheta_j\,(0|2^p\tau_{22})$, $j = 2, 3, 4$.

p=1: Let $\tau = \begin{pmatrix} \tau_{11} & \frac{1}{2} \\ \frac{1}{2} & \tau_{22} \end{pmatrix}$. Then

$$\theta\begin{bmatrix}10\\00\end{bmatrix} = \theta\begin{bmatrix}10\\01\end{bmatrix} = (2\vartheta_2\vartheta_3\widetilde{\vartheta}_3\widetilde{\vartheta}_4)^{1/2},\ \theta\begin{bmatrix}01\\10\end{bmatrix} = \theta\begin{bmatrix}01\\00\end{bmatrix} = (2\vartheta_3\vartheta_4\widetilde{\vartheta}_2\widetilde{\vartheta}_3)^{1/2},$$
$$\theta\begin{bmatrix}11\\00\end{bmatrix} = -i\theta\begin{bmatrix}11\\11\end{bmatrix} = (2\vartheta_2\vartheta_4\widetilde{\vartheta}_2\widetilde{\vartheta}_4)^{1/2},$$
$$\theta\begin{bmatrix}00\\00\end{bmatrix} = (\vartheta_3^2\widetilde{\vartheta}_3^2 + \vartheta_2^2\widetilde{\vartheta}_4^2 + \vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2},\ \theta\begin{bmatrix}00\\11\end{bmatrix} = (\vartheta_3^2\widetilde{\vartheta}_3^2 - \vartheta_2^2\widetilde{\vartheta}_4^2 - \vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2},$$
$$\theta\begin{bmatrix}00\\10\end{bmatrix} = (\vartheta_3^2\widetilde{\vartheta}_3^2 - \vartheta_2^2\widetilde{\vartheta}_4^2 + \vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2},\ \theta\begin{bmatrix}00\\01\end{bmatrix} = (\vartheta_3^2\widetilde{\vartheta}_3^2 + \vartheta_2^2\widetilde{\vartheta}_4^2 - \vartheta_4^2\widetilde{\vartheta}_2^2)^{1/2},$$

$$\begin{aligned}
\theta_1\begin{bmatrix}11\\10\end{bmatrix} &= -\pi\theta\begin{bmatrix}11\\00\end{bmatrix}\vartheta_3^2, \quad \theta_2\begin{bmatrix}11\\10\end{bmatrix} = -i\pi\theta\begin{bmatrix}11\\11\end{bmatrix}\widetilde{\vartheta}_3^2,\\
\theta_1\begin{bmatrix}11\\01\end{bmatrix} &= -i\pi\theta\begin{bmatrix}11\\00\end{bmatrix}\vartheta_3^2, \quad \theta_2\begin{bmatrix}11\\01\end{bmatrix} = -\pi\theta\begin{bmatrix}11\\00\end{bmatrix}\widetilde{\vartheta}_3^2,\\
\theta_1\begin{bmatrix}01\\01\end{bmatrix} &= -i\pi\theta\begin{bmatrix}01\\00\end{bmatrix}\vartheta_2^2, \quad \theta_2\begin{bmatrix}01\\01\end{bmatrix} = -\pi\theta\begin{bmatrix}01\\01\end{bmatrix}\widetilde{\vartheta}_4^2,\\
\theta_1\begin{bmatrix}01\\11\end{bmatrix} &= i\pi\theta\begin{bmatrix}01\\10\end{bmatrix}\vartheta_2^2, \quad \theta_2\begin{bmatrix}01\\11\end{bmatrix} = -\pi\theta\begin{bmatrix}01\\10\end{bmatrix}\widetilde{\vartheta}_4^2,\\
\theta_1\begin{bmatrix}10\\11\end{bmatrix} &= -\pi\theta\begin{bmatrix}10\\01\end{bmatrix}\vartheta_4^2, \quad \theta_2\begin{bmatrix}10\\11\end{bmatrix} = i\pi\theta\begin{bmatrix}10\\01\end{bmatrix}\widetilde{\vartheta}_2^2,\\
\theta_1\begin{bmatrix}10\\10\end{bmatrix} &= -\pi\theta\begin{bmatrix}10\\00\end{bmatrix}\vartheta_4^2, \quad \theta_2\begin{bmatrix}10\\10\end{bmatrix} = -i\pi\theta\begin{bmatrix}10\\00\end{bmatrix}\widetilde{\vartheta}_2^2.
\end{aligned}$$

p=2: Let $\tau = \begin{pmatrix} \tau_{11} & \frac{1}{4} \\ \frac{1}{4} & \tau_{22} \end{pmatrix}$ and denote $X = \vartheta_3\widetilde{\vartheta}_3$, $Y = \vartheta_2\widetilde{\vartheta}_4$, $Z = \vartheta_4\widetilde{\vartheta}_2$, $A = -X^2+Y^2+Z^2$, $B = X^2-Y^2+Z^2$, $C = X^2+Y^2-Z^2$, $D = A+B+C$. Then the following formulae hold:

$$\begin{aligned}
&\theta\begin{bmatrix}00\\00\end{bmatrix} = X+Y+Z, \quad \theta\begin{bmatrix}00\\01\end{bmatrix} = X+Y-Z,\\
&\theta\begin{bmatrix}00\\10\end{bmatrix} = X-Y+Z, \quad \theta\begin{bmatrix}00\\11\end{bmatrix} = X-Y-Z,\\
&\theta^2\begin{bmatrix}10\\00\end{bmatrix} = 2\sqrt{2}(XY)^{1/2}(D^{1/2}+\sqrt{2}Z),\ \theta^2\begin{bmatrix}10\\01\end{bmatrix} = 2\sqrt{2}(XY)^{1/2}(D^{1/2}-\sqrt{2}Z),\\
&\theta^2\begin{bmatrix}01\\00\end{bmatrix} = 2\sqrt{2}(XZ)^{1/2}(D^{1/2}+\sqrt{2}Y),\ \theta^2\begin{bmatrix}01\\10\end{bmatrix} = 2\sqrt{2}(XZ)^{1/2}(D^{1/2}-\sqrt{2}Y),\\
&\theta^2\begin{bmatrix}11\\00\end{bmatrix} = 2\sqrt{2}(YZ)^{1/2}(D^{1/2}+\sqrt{2}X), \theta^2\begin{bmatrix}11\\11\end{bmatrix} = 2\sqrt{2}(YZ)^{1/2}(D^{1/2}-\sqrt{2}X).
\end{aligned}$$

$$\theta_1\begin{bmatrix}10\\10\end{bmatrix} = -\pi(2XY)^{1/4}(\vartheta_4^2 B^{1/2}+\sqrt{2}\vartheta_3^2 Z)(D^{1/2}+\sqrt{2}Z)^{-1/2},$$
$$\theta_2\begin{bmatrix}10\\10\end{bmatrix} = -i\pi(2XY)^{1/4}(\widetilde{\vartheta}_2^2 B^{1/2}+\sqrt{2}\widetilde{\vartheta}_3^2 Z)(D^{1/2}+\sqrt{2}Z)^{-1/2},$$
$$\theta_1\begin{bmatrix}01\\01\end{bmatrix} = -i\pi(2XZ)^{1/4}(\vartheta_2^2 C^{1/2}+\sqrt{2}\vartheta_3^2 Y)(D^{1/2}+\sqrt{2}Y)^{-1/2},$$
$$\theta_2\begin{bmatrix}01\\01\end{bmatrix} = -\pi(2XZ)^{1/4}(\widetilde{\vartheta}_4^2 C^{1/2}+\sqrt{2}\widetilde{\vartheta}_3^2 Y)(D^{1/2}+\sqrt{2}Y)^{-1/2},$$
$$\theta_1\begin{bmatrix}11\\01\end{bmatrix} = -i\pi(2ZY)^{1/4}(\vartheta_3^2 C^{1/2}+\sqrt{2}\vartheta_2^2 X)(D^{1/2}+\sqrt{2}X)^{-1/2},$$
$$\theta_2\begin{bmatrix}11\\01\end{bmatrix} = -\pi(2ZY)^{1/4}(\widetilde{\vartheta}_3^2 C^{1/2}+\sqrt{2}\widetilde{\vartheta}_4^2 X)(D^{1/2}+\sqrt{2}X)^{-1/2},$$
$$\theta_1\begin{bmatrix}11\\10\end{bmatrix} = -\pi(2ZY)^{1/4}(\vartheta_3^2 B^{1/2}+\sqrt{2}\vartheta_4^2 X)(D^{1/2}+\sqrt{2}X)^{-1/2},$$
$$\theta_2\begin{bmatrix}11\\10\end{bmatrix} = -\pi(2ZY)^{1/4}(\widetilde{\vartheta}_3^2 B^{1/2}+\sqrt{2}\widetilde{\vartheta}_4^2 X)(D^{1/2}+\sqrt{2}X)^{-1/2},$$
$$\theta_1\begin{bmatrix}10\\11\end{bmatrix} = -i\pi(2XY)^{1/4}(\vartheta_4^2 A^{1/2}-\sqrt{2}i\vartheta_2^2 Z)(D^{1/2}+\sqrt{2}Z)^{-1/2},$$
$$\theta_2\begin{bmatrix}10\\11\end{bmatrix} = -i\pi(2XY)^{1/4}(\widetilde{\vartheta}_2^2 A^{1/2}-\sqrt{2}i\widetilde{\vartheta}_3^2 Z)(D^{1/2}+\sqrt{2}Z)^{-1/2},$$
$$\theta_1\begin{bmatrix}01\\11\end{bmatrix} = \pi(2XZ)^{1/4}(\vartheta_2^2 A^{1/2}-\sqrt{2}i\vartheta_3^2 Y)(D^{1/2}+\sqrt{2}Y)^{-1/2},$$
$$\theta_2\begin{bmatrix}01\\11\end{bmatrix} = -i\pi(2XZ)^{1/4}(\widetilde{\vartheta}_4^2 A^{1/2}-\sqrt{2}i\widetilde{\vartheta}_3^2 Y)(D^{1/2}+\sqrt{2}Y)^{-1/2}.$$

References

[1] H Airault, H P McKean, and J Moser. Rational and elliptic solutions of the KdV equation and a related many-body problem. *Comm.Pure and Appl. Math.*, 30:94–148, 1977.

[2] H Bateman and A Erdelyi. *Higher Transcendental Functions.* Volume 2, McGraw-Hill, New York, 1955.

[3] E D Belokolos, A I Bobenko, V Z Enolskii, A R Its, and V B Matveev. *Algebraic-geometrical Methods in the Theory of Integrable Equations.* Springer, Berlin, 1994.

[4] E D Belokolos and V Z Enolskii. Isospectral deformations of elliptic potentials. *Russian Math. Surveys*, 44:5:155–156, 1989.

[5] K J Blow, N J Doran, and D Wood. Polarization instabilities for solitons in birefringent fibers. *Opt. Lett.*, 12:202–204, 1987.

[6] O Bolza. Ueber die Reduction hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische durch eine Transformation vierten Grades. *Math. Ann.*, 28:47–496, 1887.

[7] D N Christodoulides. Black and white vector solitons in weakly birefringent optical fibers. *Phys.Lett.A*, 132:451–452, 1988.

[8] D N Christodoulides and R I Joseph. Vector solitons in birefringent nonlinear dispersive media. *Opt. Lett.*, 13:53–55, 1988.

[9] R J Dowling. Stability of solitary waves in a nonlinear birefringent optical fiber. *Physical Review A*, 42:5553–5560, 1990.

[10] J C Eilbeck and V Z Enolskii. Elliptic Baker–Akhiezer functions and an application to an integrable dynamical system. *J. Math. Phys.*, 35(3):1192–1201, 1994.

[11] J C Eilbeck, V Z Enolskii, V B Kuznetsov, and D V Leykin. Linear r-matrix algebra for systems separable in parabolic coordinates. *Phys.Lett.A*, 180:208–214, 1993.

[12] J C Eilbeck, V Z Enolskii, V B Kuznetsov, and A V Tsiganov. Linear r-matrix algebra for classical separable systems. *J. Phys. A*, 27:567–578, 1994.

[13] J C Eilbeck, P S Lomdahl, and A C Scott. The discrete self-trapping equation. *Physica D*, 16:318–338, 1985.

[14] V Z Enolskii and J C Eilbeck. On the two-gap locus for the elliptic Calogero-Moser model. 1994. Preprint hep-th/9404042.

[15] V Z Enolskii and N A Kostov. On the geometry of elliptic solitons. *Acta Applicandae Mathematicae*, 1994.

[16] M Florjanczyk and R Tremblay. Periodic and solitary waves in bimodal optical fibers. *Phys.Lett.A*, 141:34–36, 1989.

[17] J Hietarinta. Direct method for the search of the second invariant. *Physics Reports*, 147:87–154, 1987.

[18] M F Jorgensen and P L Christiansen. Hamiltonian structure for a modified Discrete Self-trapping Equation. *Chaos, Solitons, and Fractals*, 4:217–225, 1994.

[19] N A Kostov. Quasi-periodical solutions of the integrable dynamical systems related to Hill's equation. *Lett.Math.Phys.*, 17:95–104, 1989.

[20] N A Kostov and M Uzunov. New kinds of periodical waves in birefringent optical fibers. *Opt.Commun.*, 89:389–392, 1992.

[21] A Krazer and W Wirtinger. *Abelsche Funktionen und allgemeine Thetafunctionen*, pages 603–882. Teübner, 1915.

[22] N Manganaro and D F Parker. Similarity reductions for variable-coefficient coupled nonlinear Schrödinger-equations. *J. Phys. A*, 26: 4093–4106, 1993.

[23] H P McKean and P van Moerbeke. The spectrum of Hill's equation. *Invent. Math*, 30:217–274, 1977.

[24] C R Menyuk. Nonlinear pulse-propagation in birefringent optical fibers. *IEEE J.Quan.Electron.*, 23:174–176, 1988.

[25] D Mumford. *Tata lectures on theta, Vol.1, Vol.2.* Birkhäuser, Boston, 1983, 1984.

[26] S P Novikov. The periodic problem for Korteweg de Vries equation. *Funct. Analiz i ego Prilozen.*, 8:54–66, 1974.

[27] V Ravoson, A Ramani, and B Gramaticos. Generalized separability for a Hamiltonian with nonseparable quartic potential. *Phys. Lett. A*, 1994.

[28] G Rosenhain. Abhandlung uber die Funktionen zweier Variabler mit vier Perioden. *Mem. pres. l'Acad de Sci. de France. des savants*, IX:361–455, 1851.

[29] R Sahadevan, K M Tamizhmani, and M Lakshmanan. Painleve analysis and integrability of coupled non-linear Schrödinger-equations. *J.Phys.A*, 19:1783–1791, 1986.

[30] M V Tratnik and J E Sipe. Bound solitary waves in a birefringent optical fiber. *Phys.Rev.A*, 38:2011–2017, 1988.

[31] A Treibich and J-L Verdier. Solitons elliptiques. In *Special volume for 60th. anniver. of Prof. A.Grothendieck*, Birkhaüser, Boston, 1991.

[32] V E Zakharov, S V Manakov, S P Novikov, and L P Pitaevskii. *Soliton theory: inverse scattering method*. Nauka, Moscow, 1980.

[33] V E Zakharov and E I Schulman. On the integrability of the system of 2 coupled non-linear Schrödinger-equations. *Physica D*, 4:270–274, 1982.